Global maxima/minima (a.k.a. Absolute maxima/minimal)

Local minima/maxima are the minimal/maxima in the vicinity of the point. On the other hand, global minima/maxima are the minima/maxima of all.

Namely, you set a region A, and you want to find the max or min of a function on the region.

Example

- A cardboard box without a lid is to have a volume of $32\ ft^3$. What are the dimensions that minimize the amount of cardboard used?
- What are the levels of nitrogen and phosphorus to result in the best yield of an agricultural crop?

⋮

The basic way to find a global max/min is:

Theorem A global maximum/minimum is either a critical point or on the boundary of the region.

So, the main strategy to find the global max/min is:

> **Step 1.** Find the critical points.
>
> **Step 2.** Find the max/min among the values at the critical points.
>
> **Step 3** Find the max/min among the values on the boundary
>
> **Step 4.** Compare the max/min from Step 2 & Step 3 and take the largest/smallest.

This is a universal strategy that works for functions with any number of variables.

Example (One-variable case)

Find the global maximum and minimum values of $f(x) = x^3 - 3x^2 + 1$ on the region $-\frac{1}{2} \le x \le 4$.

Solution We follow the strategy.

Step 1 Find the critical points.

$f'(x) = 3x^2 - 6x = 0 \Rightarrow 3x^2 = 6x \Rightarrow x^2 = 2x \Rightarrow x = 0$ or 2.

$x=0$ and $x=2$ both appear on this region, so both are critical points.

Step 2 Find the max/min among the values at the critical points

$x=0 \Rightarrow f(0) = 1$ max $\qquad x=2 \Rightarrow f(2) = -3$ min

Step 3. Find the max/min among the values on the boundary.

The boundary of the region $-\frac{1}{2} \le x \le 4$ is $-\frac{1}{2}$ and 4.

$x = -\frac{1}{2} \Rightarrow f(-\frac{1}{2}) = -\frac{1}{8} - \frac{3}{4} + 1 = \frac{1}{8}$ min

$x = 4 \Rightarrow f(4) = 64 - 48 + 1 = 17$ max

Step 4. Compare the max/min from Step 2 & Step 3 and take the largest/smallest.

Step 2: max $x=0 \Rightarrow f(0) = 1$
min $x = 2 \Rightarrow f(2) = -3$ smallest

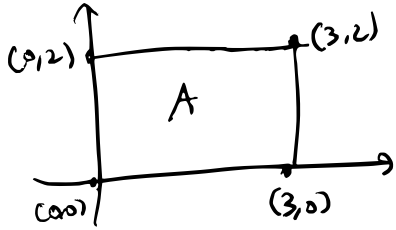
Step 3: max $x = 4 \Rightarrow f(4) = 17$ largest
min $x = -\frac{1}{2} \Rightarrow f(-\frac{1}{2}) = \frac{1}{8}$

So the global min is at $x=2 \Rightarrow f(2) = -3$
the global max is at $x = 4 \Rightarrow f(4) = 17$.

In the two variable case, it's much more difficult to deal with boundaries.

Example (Two-variable) Find the global maximum and minimum values of $f(x,y) = x^2 - 2xy + 2y$ on the region a rectangle $A = \{(x,y) \mid 0 \le x \le 3, 0 \le y \le 2\}$.

Region:



Step 1. Find the critical points.

$$f_x(x,y) = 2x-2y \qquad \Rightarrow \quad f_x(x,y)=0 \text{ means } x=y$$

$$f_y(x,y) = -2x+2 \qquad \Rightarrow \quad f_y(x,y)=0 \text{ means } x=1$$

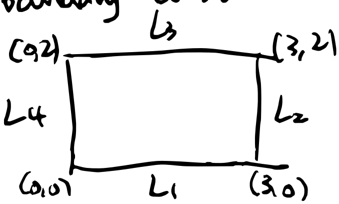
$\Rightarrow$ There is one critical point, $(1,1)$

Step 2 Find the max/min among the values at the critical points.

$f(1,1) = 1$ ← max, min.

Step 3 Find the max/min among the values on the boundary.

The boundary consists of four lines.



L_1: $0 \le x \le 3$, $y=0$

L_2: $x=3$, $0 \le y \le 2$

L_3: $0 \le x \le 3$, $y=2$

L_4: $x=0$, $0 \le y \le 2$

On each line, $f(x,y)$ looks like:

$L_1: 0 \le x \le 3,\ y=0 \Rightarrow f(x,y) = x^2-2xy+2y = x^2$

$L_2: x=3,\ 0 \le y \le 2 \Rightarrow f(x,y) = x^2-2xy+2y = 9-4y$

$L_3: 0 \le x \le 3,\ y=2 \Rightarrow f(x,y) = x^2-2xy+2y = x^2-4x+4$

$L_4: x=0,\ 0 \le y \le 2 \Rightarrow f(x,y) = 2y$

So Step 3 is split into the following sub-problems.

Step 3-1 Find the global max/min of $f(x,y)$ on L_1
Step 3-2 Find the global max/min of $f(x,y)$ on L_2
Step 3-3 Find the global max/min of $f(x,y)$ on L_3
Step 3-4 Find the global max/min of $f(x,y)$ on L_4
Step 3-5 Compare the max/min fun Step 3-1 ~ 3-4 and take the largest and the smallest.

Step 3-1. Find the global max/min of $f(x,y)$ on L_1

This amounts to finding the global max/min of $f(x)=x^2$ at $0 \le x \le 3$. This is the global max/min problem for one variable!

Step 3-1-1. Find the critical points.

$f'(x)=2x=0$ means $x=0$.

Step 3-1-2. Find the max/min of the values at the critical points

$f(0)=0$ ← max ← min

Step 3-1-3 Find the max/min of the values on the boundary.

Boundary: $x=0 \Rightarrow f(0)=0$ ← min

$x=3 \Rightarrow f(3)=9$ ← max

Step 3-1-4. Compare the max/min from Step 3-1-2 and Step 3-1-3 and take the largest & the smallest.

Critical points: $x=0 \Rightarrow f(0)=0$ ← Smallest

Boundary min: $x=0 \Rightarrow f(0)=0$ ← Smallest

max: $x=3 \Rightarrow f(3)=9$ ← largest

The global max of $f(x,y)$ at L_1 is 9,

min ... 0.

Step 3-2. Find the global max/min of $f(x,y)$ on L_2.

This amounts to finding the global max/min of $f(y)=9-4y$ at $0\le y\le 2$.

Step 3-2-1. Find the critical points.

$f'(y)=-4$ is never zero $\Rightarrow$ no critical point.

Step 3-2-2. Find the max/min of the values at the critical points.

None

Step 3-2-3 Find the max/min of the values on the boundary.

Boundary: $y=0 \Rightarrow f(0)=9$ ← max

$y=2 \Rightarrow f(2)=1$ ← min

Step 3-2-4 Compare the max/min from Step 3-2-2 and Step 3-2-3 and take the largest & the smallest.

Critical points: X

Boundary max: $f(0) = 9$ ← largest

min: $f(2) = 1$ ← smallest

The global max of $f(x,y)$ at L_2 is 9.

min ____________ 0.

Step 3-3. Find the global max/min of $f(x,y)$ on L_3

This amounts to finding the global max/min of $f(x) = x^2 - 4x + 4$ on $0 \le x \le 3$.

Step 3-3-1. Find the critical points.

$f'(x) = 2x - 4 \Rightarrow f'(2) = 0$ means $x = 2$.

Step 3-3-2 Find the max/min of the values at the critical points.

$f(2) = 0$ ← max ← min

Step 3-3-3 Find the max/min of the values on the boundary

Boundary points: $x = 0 \Rightarrow f(0) = 4$ ← max

$x = 3 \Rightarrow f(3) = 9 - 12 + 4 = 1$ ← min

Step 3-3-4 Compare the max/min from Step 3-3-2 and Step 3-3-3 and take the largest & the smallest.

Critical: $f(2) = 0$ ← smallest

Boundary min: $f(3) = 1$

Boundary max: $f(0) = 4$ ← largest

The global max of $f(x,y)$ at L_3 is 4
min ______ 0

Step 3-4. Find the global max/min of $f(x,y)$ on L_4

This amounts to finding the global max/min of $f(y) = 2y$

at $0 \le y \le 2$.

Step 3-4-1 Find the critical points

$f'(y) = 2$ is never zero $\Rightarrow$ no critical points

Step 3-4-2 Find the max/min of the values at the critical points

None.

Step 3-4-3 Find the max/min of the values on the boundary

Boundary points : $y=0 \Rightarrow f(0)=0$ ← min

$y=2 \Rightarrow f(2)=4$ ← max

Step 3-4-4 Compare the max/min from Step 3-4-2 and Step 3-4-3

and take the largest / smallest.

Critical : X

Boundary max : $f(2)=4$ ← largest

min : $f(0)=0$ ← smallest

The global max of $f(x,y)$ at L_4 is 4

min ______ 0.

Step 3-5 Take the max/min from Step 3-1 ~ 3-4 and take the largest and the smallest.

L_1: max 9
min 0

L_2: max 9
min 1

L_3: max 4
min 0

L_4: max 4
min 0

largest

smallest.

So the max on the boundary is 9.
min on the boundary is 0.

Step 4 Take the max/min from Step 2 & Step 3 and take the largest and the smallest.

Critical : 1

Boundary max : 9 ← largest
min : 0 ← smallest

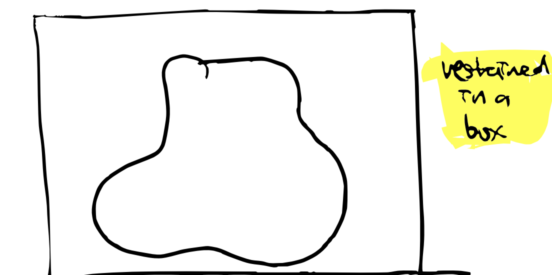
So the global max is 9, global min is 0.

Comments

1. The strategy has some loose ends, depending on how the region A looks like. The strategy will eventually work if the region is compact. Recall that compact = closed + bounded. This is why we only dealt with closed intervals in the one-variable problems.

A two-variable region is **closed** if it contains its boundary. **bounded** if it doesn't go off to infinity.

Bounded is equivalently defined as being contained in a large enough rectangle. Namely, there are upper / lower bounds for x and y coordinates.



So the region $0 \le x \le 3,\ 0 \le y \le 2$ is compact.

At this point we will only deal with compact regions, so you don't have to check it.

2. The most annoying step is Step 3, checking on the boundary step. We will learn how to approach this problem next time, and this is the method of Lagrange multipliers.

Critical points : $\nabla f = 0$

Boundary curve : Lagrange multipliers.